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Nonorientable spacetime tunneling

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Misner space is generalized to have the nonorientable topology of a Klein bottle, and it is shown that, in a classical spacetime with multiply connected space slices having such a topology, closed timelike curves are formed. Different regions on the Klein bottle surface can be distinguished which are separated by apparent horizons fixed at particular values of the two angular variables that enter the metric. Around the throat of this tunnel (which we denote a Klein bottlehole), the position of these horizons dictates an ordinary and exotic matter distribution such that, in addition to the known diverging lensing action of wormholes, a converging lensing action is also present at the mouths. Associated with this matter distribution, the accelerating version of this Klein bottlehole shows four distinct chronology horizons, each with its own nonchronal region. A calculation of the quantum vacuum fluctuations performed by using the regularized two-point Hadamard function shows that each chronology horizon nests a set of polarized hypersurfaces where the renormalized momentum-energy tensor diverges. This quantum instability can be prevented if we take the accelerating Klein bottlehole to be a generalization of a modified Misner space in which the period of the closed spatial direction is time dependent. In this case, the nonchronal regions and closed timelike curves cannot exceed a minimum size of the order the Planck scale. [S0556-2821(99)01906-2]

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I. INTRODUCTION

Most of the hitherto proposed models for spacetime tunnelings and time machines can be regarded as generalizations from Misner space [1], obtained by replacing the planes of this space with different orientable topologies, such as a sphere in wormholes [2] and a torus in ringholes [3], or by displacing by a suitable amount the period of the closed spatial direction that distinguishes Misner from Minkowskii space, as in Gott-Grant time machines [4,5]. Changing the topology of Misner space preserves time dilation as the origin for the emergence of closed timelike curves (CTCs) at sufficiently late times in the nonchronal region, and offers the possibility of obtaining exotic matter distributions near the hole throat that allowed itineraries through the tunnels along which an observer never finds any region with negative energy density [3]; hence the observer could travel safely from one mouth to the other.

It appears then of interest to investigate new kinds of tunnelings with even more complicated topology. In particular, nonorientable topologies would be especially suited, as they might give rise to sufficiently large interior regions filled with ordinary matter only. In this paper we will construct a spacetime tunnel with the topology of a Klein bottle, and discuss the properties of the possible time machines that can be built out of it.

The interest of this research would be further increased if we take into account the recent developments independently advanced by Li and Gott [6] and by González-Díaz [7], according to which Misner space becomes stable to quantum fluctuations even on the chronology (Cauchy) horizon for a convenient redefinition of its periodicity properties and, hence, of its associated vacuum. This would be a violation of chronology protection [8] which should occur in all alluded generalizations from Misner space and lend physical support to the recently considered observable effects that spacetime tunnelings would produce if they naturally existed in some sufficiently early regions of the universe [9–11]. Clearly, more complicated topologies such as that of the Klein bottle would quantitatively modify the observable predictions from cosmic wormholes and ringholes [10] and are likely to induce new observable effects. Moreover, the proposed study could also lend extra interest to the proposal that the universe was not created from nothing, but it created itseft by means of primordial CTCs [12]. However, whereas the work by Li and Gott [6] would imply the possibility for the existence of time machines and CTCs with macroscopic sizes and large travels for which the concept of chronology horizon is lost in the semiclassical treatment [13], the other stabilization procedure [7] implies a well-behaved quantization of time and chronology horizons themselves that only allows the existence of time machines and CTCs with essentially Planck size (see also Refs. [14,15] for a more balanced discussion).

Thus, the main motivation for the study of nonorientable spacetime tunnelings is related to the notion of the so-called quantum time machine [7,16] which, together with virtual black holes [17] and Euclidean wormholes [18], appears as a necessary ingredient for a consistent description of the quantum spacetime foam [19]. It seems a natural requirement that the foam should contain all possible topologies, including nonorientable ones. Actually, nonorientability may become a topological necessity if the occurrence of a nonzero minimum time and length at the Planck scale are taken to be the hallmark of quantum spacetime foam [20]. Whereas the former limit would ultimately imply the existence of causality-violating quantum time machines in the foam [19], the latter one would lead to the topological impossibility of keeping two-sideness for any two-surface lying in $\mathbf{R}^3$. Since one of the two possible sides of such surfaces can always be made topologically inaccessible by the uncertainty $\Delta x \geqslant$

Planck length in the foam, any closed two-surfaces (that is, any two-manifold) lying in the foam should necessarily be one-sided and hence nonorientable.

The paper is organized as follows. Using a given geometric *Ansatz*, in Sec. II we describe a way to obtain the static metric on some sections of the spacetime generated by a distribution of matter with the topology of a Klein bottle, and discuss the existence of apparent horizons at fixed values of the angular variables that define a nonorientably deformed toroidal geometry. Starting with the metrics obtained for the Klein bottle sections, we derive in Sec. III a spacetime that describes what we may call a Klein bottlehole, that is, a tunnel in Lorentz spacetime with the symmetry of a Klein bottle, discussing the conditions required by this tunnel to be embedded in flat space, and the characteristics of the stress-energy tensor needed to make it compatible with general relativity and the lensing actions expected to be induced in its mouths. Section IV contains a discussion of the conversion of this Klein bottlehole into time machine, i.e., into an accelerating Klein bottlehole, briefly analyzing the causal and noncausal structure of the resulting space. Also in Sec. IV is a calculation of the quantum effects implied by vacuum polarization inside the chronology horizons, following the procedure used by Kim and Thorne [21], and a brief discussion of the above spacetime construct in the case in which the period of the closed spatial direction is time dependent, for which case no polarized hypersurfaces with divergent vacuum polarization are allowed. Finally, we summarize our results in Sec. V.

II. STATIC METRIC ON THE KLEIN BOTTLE

We shall consider the gravitational field created by a distribution of matter with the symmetry of a Klein bottle, and obtain the static spacetime metric for constant surfaces possessing such a symmetry. In order to account for the nonorientable character of the Klein bottle, we shall extend the range of the angular coordinate $\varphi_1 0 \leqslant \varphi_1 \leqslant 2\pi$ on the circular axis of the orientable torus [3] to also encompass the values continuously running from 2π to 3π, while allowing the radii of the transversal section of the so-deformed torus tube and of its deformed axis to be both φ_1 dependent, with the transversal surfaces at $\varphi_1 = 3\pi$ and at $\varphi_1 = 0$ identified. In Fig. 1 we define the Cartesian coordinates on a Klein bottle. These can be written as $x = m_1 \sin\varphi_1$, $y = m_1 \cos\varphi_1$ and $z = b_1 \sin\varphi_2$ for $0 \leqslant \varphi_1 \leqslant 2\pi$, and $x = m_2 \sin\varphi_1$, $y = A_1 - C_2 - m_2 \cos\varphi_1$ and $z = b_2 \sin\varphi_2$, for $2\pi \leqslant \varphi_1 \leqslant 3\pi$, with

$$m_1 = a_1 - b_1 \cos\varphi_2, \quad n_1 = b_1 - a_1 \cos\varphi_2, \tag{2.1}$$

where $0 \leqslant \varphi_2 \leqslant 2\pi$, and we have used the *Ansatz*

$$a_1 \equiv a_1(\varphi_1) = (A_1 - C_1)\cos^2\frac{\varphi_1}{4} + C_1, \tag{2.2}$$

$$b_1 \equiv b_1(\varphi_1) = (B_1 - D_1)\cos^2\frac{\varphi_1}{4} + D_1, \tag{2.3}$$

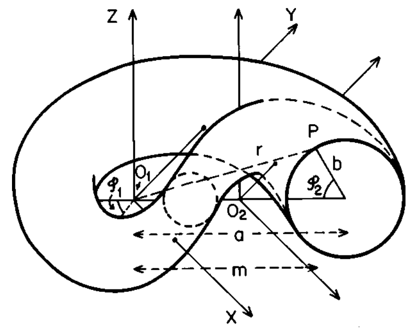

FIG. 1. Cartesian coordinates on the two-dimensional Klein bottle. Any point P on the Klein bottle surface can be labeled by parameters $a, b, \varphi_1, \varphi_2$. If P correspons to the interval $0 \leqslant \varphi_1 \leqslant 2\pi$ (as displayed on the figure), then the reference frame to fix the given point is that with origin at O_1 (with parameters a_1, b_1, m_1, r_1 in the main text), and if P corresponds to the φ_1 interval $2\pi < \varphi_1 < 3\pi$ associated with the bottle region which makes it nonorientable, then it would be given in terms of the reference frame with origin at O_2 (with parameters a_2, b_2, m_2, r_2 in the main text).

in which A_1, B_1, C_1, and D_1 are adjustable constant parameters satisfying the conditions $A_1 > C_1, B_1 > D_1, A_1 > B_1$, and $C_1 > D_1$, with a_1 and b_1 the varying radius of the circumference generated by the Klein bottle axis and that of the transversal section of the Klein bottle tube, respectively, in the region $0 \leqslant \varphi_1 \leqslant 2\pi$. For the interval $2\pi \leqslant \varphi_1 \leqslant 3\pi$, we have

$$m_2 = a_2 + b_2 \cos\varphi_2, \quad n_2 = b_2 + a_2 \cos\varphi_2, \tag{2.4}$$

for the associated *Ansatz*

$$a_2 \equiv a_2(\varphi_1) = (C_2 - A_2)\sin^2\frac{\varphi_1}{2} + A_2, \tag{2.5}$$

$$b_2 \equiv b_2(\varphi_1) = (D_2 - B_2)\sin^2\frac{\varphi_1}{2} + B_2, \tag{2.6}$$

where the conditions for the new adjustable constant parameters are $C_2 > A_2$, $D_2 > B_2$, $C_2 > D_2$, and $A_2 > B_2$, and $D_2 = B_1$, $B_2 = D_1$, and $A_1 - C_1 = A_2 + C_2$, with $A_1 - C_1 > 2A_2$.

We have, in the region $0 \leqslant \varphi_1 \leqslant 2\pi$,

$$\begin{aligned} d\Omega_1^2 &= dx^2 + dy^2 + dz^2 \\ &= \left\{ m_1^2 + \frac{1}{4}[M_1(a_1 - C_1) + N_1(b_1 - D_1)] \right\} d\varphi_1^2 + b_1^2 d\varphi_2^2 \\ &\quad - b_1\sqrt{(a_1 - C_1)(A_1 - a_1)}\sin\varphi_2 d\varphi_1 d\varphi_2, \end{aligned} \tag{2.7}$$

in which

$$M_1 = A_1 - a_1 - (B_1 - b_1)\cos\varphi_2, \tag{2.8}$$

$$N_1 = B_1 - b_1 - (A_1 - a_1)\cos\varphi_2, \tag{2.9}$$

and, in the region $2\pi \leqslant \varphi_1 \leqslant 3\pi$,

$$d\Omega_2^2 = dx^2+dy^2+dz^2 = \{m_2^2+M_2(a_2-A_2)+N_2(b_2-B_2)\}d\varphi_1^2+b_2^2 d\varphi_2^2 - 2b_2\sqrt{(a_2-A_2)(C_2-a_2)}\sin\varphi_2 d\varphi_1 d\varphi_2, \quad (2.10)$$

where

$$M_2=C_2-a_2+(D_2-b_2)\cos\varphi_2, \quad (2.11)$$

$$N_2=D_2-b_2+(C_2-a_2)\cos\varphi_2. \quad (2.12)$$

We can assume then for the static metric corresponding to a distribution of matter with the symmetry of a Klein bottle the general expression

$$ds^2=-e^{\Phi}dt^2+e^{\Psi}dr^2 + \theta(2\pi-\varphi_1)d\Omega_1^2+\theta(\varphi_1-2\pi)d\Omega_2^2, \quad (2.13)$$

in which the $\theta(x)$'s are the step Heaviside function, with $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$,

$$r=r_1=\sqrt{a_1^2+b_1^2-2a_1b_1\cos\varphi_2}, \quad (2.14)$$

for $0\leqslant\varphi_1\leqslant 2\pi$,

$$r=r_2=\sqrt{a_2^2+b_2^2+2a_2b_2\cos\varphi_2}, \quad (2.15)$$

for $2\pi\leqslant\varphi_1\leqslant 3\pi$, and Φ and Ψ will generally depend on t and r in the respective interval of φ_1.

Denoting by x^0, x^1,x^2, and x^3, respectively, the coordinates on the Klein bottle ct, r,φ_1, and φ_2, we have, for the nonzero components of the metric tensor, $g_{00}=-e^{\Phi}$, $g_{11}=e^{\Psi}$,

$$g_{22}=\left\{m_1^2+\frac{1}{4}[M_1(a_1-C_1)+N_1(b_1-D_1)]\right\}\theta(2\pi-\varphi_1) +[m_2^2+M_2(a_2-A_2)+N_2(b_2-B_2)]\theta(\varphi_1-2\pi),$$

$$g_{23}=-[b_1\sqrt{(a_1-C_1)(A_1-a_1)}\theta(2\pi-\varphi_1) +2b_2\sqrt{(a_2-A_2)(C_2-a_2)}\theta(\varphi_1-2\pi)]\sin\varphi_2,$$

$$g_{33}=b_1^2\theta(2\pi-\varphi_1)+b_2^2\theta(\varphi_1-2\pi).$$

Using then the expressions for the derivatives that result from our previous definitions and *Anästze*, i.e.,

$$\frac{dm_i}{dr_i}=\frac{r_i}{a_i}, \quad \frac{dn_i}{dr_i}=\frac{r_i}{b_i}, \quad \frac{n_i}{m_i}=\frac{a_i}{b_i}, \quad (2.16)$$

$$\frac{dm_i}{d\varphi_2}=-(-1)^i b_i\sin\varphi_2, \quad \frac{dn_i}{d\varphi_2}=-(-1)^i a_i\sin\varphi_2, \quad (2.17)$$

$$\frac{da_i}{dr_i}=\frac{r_i}{m_i}, \quad \frac{db_i}{dr_i}=\frac{r_i}{n_i}, \quad (2.18)$$

$$\frac{da_i}{d\varphi_1}=\frac{i}{2}(-1)^i\sqrt{(a_i-C_i)(A_i-a_i)}, \quad (2.19)$$

$$\frac{db_i}{d\varphi_1}=\frac{i}{2}(-1)^i\sqrt{(b_i-D_i)(B_i-b_i)}, \quad (2.20)$$

where $i=1,2$, one can calculate the components of the affine connection and hence those of the Ricci curvature tensor. From the resulting Einstein equations that correspond to the gravitational field of a matter distribution with the symmetry of the Klein bottle used in this paper, one finally obtains

$$\frac{8\pi G}{c^4}(T_0^0-T_1^1)e^{\Psi} = -\frac{1}{4}(\Psi'+\Phi')P+\frac{1}{2}P'+\frac{1}{4}P^2+\frac{1}{2}Q, \quad (2.21)$$

$$\frac{8\pi G}{c^4}(T_2^2-T_3^3)e^{\Psi} = \frac{1}{4}(\Psi'-\Phi')R-\frac{1}{2}R'+\frac{1}{2}(\ln g_{23})'R-\frac{1}{2}RS+Te^{\Psi}, \quad (2.22)$$

where the prime denotes derivative with respect to the corresponding coordinate r_i, and

$$P=(\ln g_{22})'+2(\ln g_{23})'+(\ln g_{33})', \quad (2.23)$$

$$Q=[g^{22}g^{33}+(g^{23})^2][g'_{22}g'_{33}-(g'_{23})^2], \quad (2.24)$$

$$R=(\ln g_{22})'-(\ln g_{33})', \quad S=(\ln g_{22})'+(\ln g_{33})', \quad (2.25)$$

$$T=\frac{1}{2}g^{33}\left(\frac{dg^{22}}{d\varphi_2}\frac{dg_{23}}{d\varphi_1}-\frac{dg^{22}}{d\varphi_1}\frac{dg_{23}}{d\varphi_2}+2\frac{dg^{23}}{d\varphi_2}\frac{dg_{33}}{d\varphi_1}\right) -g^{22}\left(\frac{1}{2}\frac{dg^{33}}{d\varphi_1}\frac{dg_{23}}{d\varphi_2}+\frac{dg^{23}}{d\varphi_1}\frac{dg_{22}}{d\varphi_2}\right) -\frac{1}{2}g^{22}(g^{23})^2\left[\left(\frac{dg_{22}}{d\varphi_2}\right)^2+\frac{dg_{22}}{d\varphi_1}\frac{dg_{33}}{d\varphi_1}\right] +\frac{1}{2}g^{33}(g^{23})^2\left(\frac{dg_{33}}{d\varphi_1}\right)^2, \quad (2.26)$$

with the expressions for the metric tensor components as given above.

General solutions to these equations look quite complicated, even for the vacuum case, $T_\mu^\nu=0$. Nevertheless, one can still derive solutions to Eqs. (2.21) and (2.22) on two-dimensional sections in the vacuum case. Thus, on the sections $\varphi_1=$const or on the sections $\varphi_2=$const we can obtain solutions in closed form and investigate the possible horizons which can appear along the angular coordinates. Let us first consider constant-φ_1 sections of the Klein bottle. In this case, we have the solution

$$\Phi=0, \quad \Psi=2\ln\left[\frac{4b_i r_i}{r_i^2-(a_i^2-b_i^2)}\right], \quad i=1,2, \tag{2.27}$$

where we have chosen the integration constant to be zero and a_i and b_i are constant.

Solution (2.27) is defined for $0\leqslant t\leqslant\infty$, $0\leqslant\varphi_2\leqslant 2\pi$, and $a_i-b_i\leqslant r_i\leqslant a_i+b_i$, on constant-$\varphi_1$ sections which are fixed either on $0\leqslant\varphi_1\leqslant 2\pi$ when $i=1$ or on $2\pi\leqslant\varphi_1\leqslant 3\pi$ when $i=2$, and describes the spacetime geometry on the corresponding φ_1 section of a Klein bottle with constant a_i and b_i radii, generated by varying angle φ_2 only. The variation of the metric with angle φ_2 is of interest in order to determine the position of angular horizons. On $\varphi_2=\pi, r_1=r_{1max}=a_1+b_1$ and $r_2=r_{2min}=a_2-b_2$, the metric becomes singular. As φ_2 decreases from π to $\varphi_2=\varphi_2^c=\arccos(b_1/a_1), r_{1max}$ decreases to $\sqrt{a_1^2-b_1^2}$ and r_{2min} increases up to $\sqrt{a_2^2-b_2^2}$, where the metric is singular again. It is also singular on $\varphi_2=0$, where $r_1=r_{1min}=a_1-b_1$ and $r_2=r_{2\max}=a_2+b_2$, and on $\varphi_2=2\pi-\varphi_2^c$. All of these singularities are not true singularities, but arise only from the choice of coordinates. They are also present in the static metric on a two-torus, which is the orientable topology that directly corresponds to that of a Klein bottle.

If we introduce the new coordinates

$$U_i=t+2b_i\ln[r_i^2-(a_i^2-b_i^2)], \tag{2.28}$$

$$V_i=t-2b_i\ln[r_i^2-(a_i^2-b_i^2)], \tag{2.29}$$

then the metric transforms into

$$ds^2=dU_i dV_i+\theta(2\pi-\varphi_1)d\Omega_1^2+\theta(\varphi_1-2\pi)d\Omega_2^2, \tag{2.30}$$

where

$$r_i^2=a_i^2-b_i^2+\exp\left(\frac{U_i-V_i}{4b_i}\right). \tag{2.31}$$

Metric (2.30) is in fact regular on the above horizons.

Consider now constant-φ_2 sections of the Klein bottle. In this case, we get the closed form vacuum solution $\Phi=0$ and

$$\Psi=\ln\left\{\frac{\frac{2m_i}{a_i}+\frac{i^2}{4}\left(-4+\frac{\alpha_i}{a_i}+\frac{\beta_i}{b_i}+\frac{m_i^{(0)}}{m_i}+\frac{n_i^{(0)}}{n_i}\right)r_i^2}{m_i^2+\frac{i^2}{4}[M_i(a_i-\alpha_i)+N_i(b_i-\beta_i)]}\right\}, \tag{2.32}$$

where $i=1,2$, $\alpha_{1,2}=C_1, A_2$, $\beta_{1,2}=D_1$, B_2, and $m_i^{(0)}$ and $n_i^{(0)}$ are constant,

$$m_i^{(0)}=\delta_i+(-1)^i\gamma_i\cos\varphi_2, \quad n_i^{(0)}=\gamma_i+(-1)^i\delta_i\cos\varphi_2, \tag{2.33}$$

with $\delta_{1,2}=A_1, C_2$ and $\gamma_{1,2}=B_1, D_2$.

We note that there is no singularity on the surfaces $r_i=\sqrt{a_i^2-b_i^2}$ since both the numerator and the denominator of the g_{11} component of the metric tensor corresponding to solution (2.32) go to zero on such surfaces. On the region $2\pi\geqslant\varphi_1\geqslant 0$ singularities would appear at values of angle φ_1 given by

$$\varphi_1=4\arccos\chi_1, \tag{2.34}$$

where

$$\chi_1=\sqrt{\frac{8m_1^1\Delta m_1+W_1+4\sqrt{W_1[m_1^1\Delta m_1+W_1/16+(m_1^1)^2]}}{2[W_1-4(\Delta m_1)^2]}}, \tag{2.35}$$

in which

$$m_1^1=C_1-D_1\cos\varphi_2, \quad \Delta m_1=m_1^{(0)}-m_1^1, \tag{2.36}$$

$$n_1^1=D_1-C_1\cos\varphi_2, \quad \Delta n_1=n_1^{(0)}-n_1^1, \tag{2.37}$$

$$W_1=\Delta m_1(A_1-C_1)+\Delta n_1(B_1-D_1). \tag{2.38}$$

It can now be checked that a singularity can only appear on this region whenever $B_1-D_1>A_1-C_1$, i.e., provided the radius b_1 of the Klein bottle decreases more rapidly than the internal radius a_1 does as one approaches $\varphi_1=2\pi$.

As for region $2\pi\leqslant\varphi_1\leqslant 3\pi$, singularities may be present whenever

$$\varphi_1=4\arcsin\chi_2, \tag{2.39}$$

where

$$\chi_2=\sqrt{\frac{2m_2^1\Delta m_2+W_2\pm 2\sqrt{W_2[m_2^1\Delta m_2+W_2/4+(m_2^1)^2]}}{2[W_2-(\Delta m_2)^2]}}, \tag{2.40}$$

in which

$$m_2^1=A_2+B_2\cos\varphi_2, \quad \Delta m_2=m_2^{(0)}-m_2^1, \tag{2.41}$$

$$n_2^1=B_2+A_2\cos\varphi_2, \quad \Delta n_2=n_2^{(0)}-n_2^1, \tag{2.42}$$

$$W_2=\Delta m_2(C_2-A_2)+\Delta n_2(D_2-B_2). \tag{2.43}$$

On this region the existence of singularities does not impose any constraints on the values of the parameters that define the geometry of the Klein bottle, unless that A_2 must vanish on the singularities. Then, singularities would appear for any values of C_2, B_2, and D_2, provided $A_2=0$, satisfying the conditions that follow the *Ansätze* (2.5) and (2.6) only on the extreme, critical values $\varphi_1=2\pi$ and $\varphi_1=3\pi$.

III. SPACETIME OF A KLEIN BOTTLEHOLE

More complicated, but similar to what happens in ringholes [3], the creation of traversible nonorientable holes respecting Einstein equations, so that classical general relativity be valid everywhere, should be accompanied by the formation at late times of CTCs in some nonchronal spacetime domains and by violation of the averaged weak energy condition [2,22] only on some restricted, classically forbidden regions bounded by the angular horizons dealt with in the previous section. On the other hand, one should also expect that violation of the energy condition would not ultimately induce any divergences of either the expectation value of a propagating scalar field squared or the renormalized stress-energy tensor if, besides replacing the planes of Misner space for Klein bottles, one takes the period associated with the closed spatial dimension to be time dependent and given by $a=2\pi t^2$ [7]. The result would be a quantically stable spacetime tunneling possessing CTCs only at the Planck scale. If, moreover, we would take that period to be $a=2\pi$ and redefine the vacuum consequently as Li and Gott have recently done [6], the resulting nonchronal region will, in principle, not possess instabilities anywhere, too, and would give rise to CTCs which are not restricted in size. In this case, however, the concept of a chronology horizon can be argued to be lost [13].

A static nonorientable hole having the topology of a Klein bottle would be traversible if a two-Klein-bottle surrounding one of its mouths where spacetime is nearly flat can be regarded as an outer trapped surface to an observer looking through the hole from the other mouth [2,23]. The static spacetime metric for one such single, traversible Klein bottlehole may, in principle, be written in the form

$$ds^2=-dt^2+\theta(2\pi-\varphi_1)\left[\left(\frac{n_1(l_1)}{r_1(l_1)}\right)^2 dl_1^2+d\Omega_1^2(l_1)\right]$$
$$+\theta(\varphi_1-2\pi)\left[\left(\frac{n_2(l_2)}{r_2(l_2)}\right)^2 dl_2^2+d\Omega_2^2(l_2)\right], \tag{3.1}$$

where $-\infty<t<+\infty$, $-\infty<l_i<+\infty$, the $d\Omega_i^2$'s are as given by Eqs. (2.7) and (2.10), with b_i replaced for $\sqrt{l_i^2+b_{0i}^2}$, and l_i the proper radial distance of each transversal section of the Klein bottle on the respective φ_i interval for $i;b_{0i}$ is as given by Eqs. (2.3) and (2.6) for constant parameters ajusted to the radius of the double throat of the Klein bottle that occurs at $l_i=0$. Consequently,

$$m_i(l_i)=a_i+(-1)^i\sqrt{l_i^2+b_{0i}^2}\cos\varphi_2, \tag{3.2}$$

$$n_i(l_i)=\sqrt{l_i^2+b_{0i}^2}+(-1)^i a_i\cos\varphi_2, \tag{3.3}$$

$$r_i(l_i)=\sqrt{a_i^2+l_i^2+b_{0i}^2+2(-1)^i\sqrt{l_i^2+b_{0i}^2}a_i\cos\varphi_2}. \tag{3.4}$$

Metric (3.1) would give us a particularly simple example of a traversible nonorientable hole which can be readily generalized. Thus, one can convert Eq. (3.1) into the more general static Klein bottlehole metric

$$ds^2=-e^{2\Phi}dt^2+\theta(2\pi-\varphi_1)\left(\frac{dr_1^2}{1-K(b_1)/b_1}+d\Omega_1^2\right)$$
$$+\theta(\varphi_1-2\pi)\left(\frac{dr_2^2}{1-K(b_2)/b_2}+d\Omega_2^2\right)$$
$$=-e^{2\Phi}dt^2+\theta(2\pi-\varphi_1)\left[\left(\frac{n_1}{r_1}\right)^2 dl_1^2+d\Omega_1^2\right]$$
$$+\theta(\varphi_1-2\pi)\left[\left(\frac{n_2}{r_2}\right)^2 dl_2^2+d\Omega_2^2\right] \tag{3.5}$$

if we let $\Phi=0,K(b_i)=b_{0i}^2/b_i$, and $l_i=\pm\sqrt{b_i^2-b_{0i}^2}$, where the minus sign applies on the left side of the throat and the plus sign applies on the right side [2]. Φ will generally be given now as a function of the mass of the nonorientable Klein bottlehole and the geometric parameters that determine it.

The metric (3.5) can be regarded as a generalization to Klein bottle symmetry from the static metric of a toroidal ringhole and, hence, from that of a spherical wormhole metric. One can obtain the line element for a ringhole spacetime from Eq. (3.5) by using the set of parameters $A_1=C_1\neq 0$, $B_1=D_1\neq 0$, $A_2=C_2=B_2=D_2=0$, and from the ringhole metric we obtain the line element of a spherical wormhole by using the transformations $a\to 0,\varphi_2\to\theta+\pi/2$ [3]. Thus, we obtain first, for the ringhole metric [3],

$$ds^2=-e^{2\Phi}dt^2+\left(\frac{n}{r}\right)^2 dl^2+m^2d\varphi_1^2+b^2d\varphi_2^2 \tag{3.6}$$

and then, for the wormhole metric [2],

$$ds^2=-e^{2\Phi}dt^2+dl^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{3.7}$$

However, in order for the metric (3.5) to represent tunneling through a traversible nonorientable Klein bottlehole, while satisfying Einstein equations for a convenient stress-energy tensor, it must be embeddible in a three-dimensional Euclidean space at a fixed time t. What one should actually consider for such a purpose is a three-geometry respecting the symmetry of the Klein bottle and satisfying $a_i\geqslant b_i>l_i$, visualizing then the given slice as removed from the spacetime of the nonorientable Klein bottlehole and embedded in three-dimensional Euclidean space. As it stands, metric (3.5) is not the metric required for such an embedding, at least if we take for the embedding space a space with cylindrical coordinates z,r,ϕ:

$$ds^2=dz^2+dr^2+r^2d\phi^2. \tag{3.8}$$

Nevertheless, since r_i and φ_1 are not independent of each other, one can always convert Eq. (3.5) into a metrical form which is embeddible in the cylindrical space (3.8). In order to do that conversion, we first obtain the formula that expresses the way in which r_i varies with φ_1, i.e.,

$$\frac{dr_i}{d\varphi_i} \equiv Q(i) = -\frac{[m_i(A_i - C_i) + n_i(B_i - D_i)]\sin(i\varphi_1/2)}{2ir_i}. \tag{3.9}$$

Hence,

$$ds^2 = -e^{2\Phi}dt^2 + \theta(2\pi - \varphi_1)\left(\frac{c(1)dr_1^2}{1 - b_{01}^2/b_1^2} + d(1)Q(1)dr_1 d\varphi_1 + d\Omega_1^2\right) + \theta(\varphi_1 - 2\pi) \times \left(\frac{c(2)dr_2^2}{1 - b_{02}^2/b_2^2} + d(2)Q(2)dr_2 d\varphi_1 + d\Omega_2^2\right), \tag{3.10}$$

with $c(i) + d(i) = 1$.

Taking $dz = (dz/dr_1)dr_1 + (dz/d\varphi_1)d\varphi_1$ for $\varphi_1 \leqslant 2\pi$ or $dz = (dz/dr_2)dr_2 + (dz/d\varphi_1)d\varphi_1$ for $\varphi_1 > 2\pi$, one can obtain, for any value of the coordinate φ_2,

$$c(i) = 1 + 2\left(1 - \frac{b_{0i}^2}{b_i^2}\right) - 2\sqrt{1 - \frac{b_{0i}^2}{b_i^2}}. \tag{3.11}$$

Therefore, the metric for the nonorientable Klein bottlehole which is embeddible in flat space is described by Eq. (3.10), with $c(i)$ given by Eq. (3.11) and $d(i) = 1 - c(i)$. Using these coefficients, metric (3.8) will be the same as metric (3.10) for constant values of φ_2 if we identify the coordinates r, ϕ of the embedding space with either the coordinates r_1, φ_1, for $\varphi_1 \leqslant 2\pi$, or the coordinates r_2, φ_1, for $\varphi_1 > 2\pi$, and if we require the function z to satisfy

$$\frac{dz}{dr_i} = 1 + \left(1 - \frac{b_{0i}^2}{b_i^2}\right)^{-1} - 2\left(1 - \frac{b_{0i}^2}{b_i^2}\right)^{-(1/2)}, \tag{3.12}$$

for any value of φ_1,

$$\frac{dz}{d\varphi_1} = \frac{1}{2}\sqrt{[R(\varphi_2)_1 - r_1][r_1 - \rho(\varphi_2)_1]}, \tag{3.13}$$

for $\varphi_1 \leqslant 2\pi$, and

$$\frac{dz}{d\varphi_1} = \sqrt{[R(\varphi_2)_2 - r_2][r_2 - \rho(\varphi_2)_2]}, \tag{3.14}$$

for $\varphi_1 > 2\pi$, where

$$R(\varphi_2)_1 = A_1 - B_1 \cos\varphi_2, \quad \rho(\varphi_2)_1 = C_1 - D_1 \cos\varphi_2 \tag{3.15}$$

and

$$R(\varphi_2)_2 = C_2 + D_2 \cos\varphi_2, \quad \rho(\varphi_2)_2 = A_2 + B_2 \cos\varphi_2. \tag{3.16}$$

From these expressions and the requirement that nonorientable Klein bottleholes be connectible to asymptotically flat spacetime, one can deduce how the embeddible surfaces would flare at or around the hole throat. Thus, from Eq. (3.12) one obtains

$$\frac{d^2r}{dz^2} = \frac{b_{0i}^2 r_i}{b_i^3 n_i}\left(\frac{1}{\sqrt{1 - b_{0i}^2/b_i^2}} - 1\right) \times \left(1 + \frac{1}{1 - b_{0i}^2/b_i^2} - \frac{2}{\sqrt{1 - b_{0i}^2/b_i^2}}\right)^{-(7/2)}, \tag{3.17}$$

which is positive for $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c = \arctan(b_i/a_i)$ and negative for $-\varphi_2^c < \varphi_2 < \varphi_2^c$. Thus, exactly as happens in the case of toroidal ringholes [3], the embedding surface flares outward for $d^2r/dz^2 > 0$ and flares inward for $d^2r/dz^2 < 0$.

To investigate how the embedding surface flares at or around the throat as the angle φ_1 is varied, we have to distinguish two cases. The first one corresponds to condition (3.13), from which we can get

$$\frac{d^2\varphi_1}{dz^2} = \frac{[-2r_1 + R(\varphi_2)_1 + \rho(\varphi_2)_1][R(\varphi_2)_1 - \rho(\varphi_2)_1]\sin\varphi_1/2}{2\{[R(\varphi_2)_1 - r_1][r_1 - \rho(\varphi_2)_1]\}^2}. \tag{3.18}$$

Since $a_1 > b_1$ for $0 \leqslant \varphi_1 \leqslant 2\pi$, the sign of the right-hand side (RHS) of Eq. (3.18) will be fixed by the sign of the quantity in the first brackets in its numerator. One obtains that Eq. (3.18) vanishes for $\varphi_1 = \varphi_1^2 = \pi$ and becomes negative for $\varphi_1 < \pi$, for which angular values the embedding surface flares toward larger values of the radius b_1 and negative for $\varphi_1 > \pi$, on which region the embedding surface flares toward smaller values of b_1.

The second case comes from condition (3.14). Here we get

$$\frac{d^2\varphi_1}{dz^2}=-\frac{[-2r_2+R(\varphi_2)_2+\rho(\varphi_2)_2][R(\varphi_2)_2-\rho(\varphi_2)_2]\sin\varphi_1}{4\{[R(\varphi_2)_2-r_2][r_2-\rho(\varphi_2)_2]\}^2}. \tag{3.19}$$

The critical value of φ_1 becomes then $\varphi_1=\varphi_1^c=5\pi/2$. Again it is the quantity in the first brackets in the numerator of the RHS of Eq. (3.19) which determines the sign of this equation. For $\varphi_1<5\pi/2$, that sign is negative so that the embedding surface flares toward smaller values of b_2, while it becomes positive for $\varphi_1>5\pi/2$, where the embedding surface flares toward larger values of b_2.

On the other hand, from the Einstein equations (2.21), on the region $0\leqslant\varphi_1\leqslant2\pi$, we can obtain, for the metric components of metric (3.10) with $\Phi=0$,

$$\frac{\frac{8m_1}{a_1}-\left(\frac{A_1-C_1}{a_1}+\frac{B_1-D_1}{b_1}\right)\cos^2\frac{\varphi_1}{4}+\frac{m_1^{(0)}}{m_1}+\frac{n_1^{(0)}}{n_1}-2}{4\left\{n_1m_1^2+\frac{1}{4}[M_1(A_1-C_1)+N_1(B_1-D_1)]\cos^2(\varphi_1/4)\right\}}$$
$$+\frac{2(1+\sin\varphi_2)}{n_1b_1}-\frac{4\cos(\varphi_1/2)\sin\varphi_2}{(A_1-C_1)m_1\sin^2(\varphi_1/2)}$$
$$=Y_1(\varphi_1,\varphi_2)=\frac{8\pi G}{c^4r_1}(T_0^0-T_1^1). \tag{3.20}$$

The stress-energy tensor components T_0^0 and T_1^1 in Eq. (3.20) cannot be directly expressed in terms of, respectively, an energy density ρ and a tension per unit area σ for the symmetry associated with a Klein bottle. Such as happened for a toroidal symmetry [3], here the tensor components T_μ^ν will also depend explicitly on r_1, whereas ρ and σ should be defined as a function of the (nonorientable) normal to the surface element on the Klein bottle, on the region where $0\leqslant\varphi_1\leqslant2\pi$, along the direction determined by the radius b_1. Since $db_1/dr_1=r_1/n_1$, in the neighborhood of the throat where $b_1\simeq b_{01}$, we must have

$$\rho c^2-\sigma=\left(\frac{n_1}{r_1}\right)^3(T_0^0-T_1^1)\simeq\frac{c^4b_{01}^2n_1^2}{16\pi Gb_1^3r_1^2}Y_1(\varphi_1,\varphi_2). \tag{3.21}$$

Now, since the factor in front of Y_1 in Eq. (3.21) is positive definite, it follows that

$$\mathrm{sgn}[\rho c^2-\sigma]\,\mathrm{sgn}\,Y_1(\varphi_1,\varphi_2), \tag{3.22}$$

at or near the nonorientable hole throat. An analysis of the function $Y_1(\varphi_1,\varphi_2)$ indicates that $\rho c^2-\sigma$ will be negative for small values of the involved angles φ_1 and φ_2, and positive as one approaches either $\varphi_1=2\pi$ or $\varphi_2=\pi$. There will be then intermediate critical values for these angles at which $Y_1=0$. These critical values will depend on the values of the adjustable parameters that define the radii a_1 and b_1. A similar analysis can be made for the region $2\pi\leqslant\varphi_1\leqslant3\pi$, at or near the throat, which allows us to conclude that the new function Y_2 will be positive for values of φ_1 close to 2π, and becomes negative as φ_1 approaches 3π, having the same behavior as Y_1 with respect to variation with angle φ_2. All of these results have been obtained for the specific metric where $\Phi=0$, but it is easy to check that they are still valid for any other value of Φ, provided it is everywhere finite. It follows that for an observer moving through the Klein bottlehole's throat with a sufficiently large speed, $\gamma\gg1$, the energy density $\gamma^2(\rho c^2-\sigma)+\sigma$ will take on positive or negative values depending on the specific combination of values he chooses for φ_1, φ_2, A_i, B_i, C_i, and D_i.

One would expect lensing effects to occur on the mouths of the nonorientable Klein bottlehole with respect to a bundle of light rays, at or near the throat, coming from the distribution of positive and negative values for the energy density; i.e., the mouths would act like a diverging lens for world lines along the values of the coordinates, at or near the throat, which correspond to negative energy density, and like a converging lens for world lines passing through regions with positive energy density. Thus, at or near the throat of the Klein bottlehole, one would expect diverging lens effects to tend to be concentrated onto those values of φ_1 for which the radius of the transversal section of the Klein bottle becomes larger and on the regions described by values of φ_2 which tend to concentrate about $\varphi_2=\pi$. The exact relative extend of such regions will ultimately depend on the precise values used for the constant parameters that define the radii a_i and b_i. Actually, in order to acertain with full accuracy which regions around the throat behave like a lens a way or another, one should consider the null-ray propagation governed by the integral of the stress-energy tensor. For the mouths to defocus a bundle of rays, such an integral

$$\int_{l_i^1}^{\infty}dl_ie^{-\Phi}(\rho c^2-\sigma)$$

must turn out to be negative for any $l_i^1<0$ and positive if the mouths focus the rays. By using expressions such as Eq. (3.21), one can check the above conclusions, for any Φ which is everywhere finite.

IV. NONORIENTABLE TIME MACHINE AND VACUUM FLUCTUATIONS

The nonorientable Klein bottlehole considered in Sec. III can be viewed as a generalization from Misner space, obtained by replacing the identified flat planes of this space for identified Klein bottles. It actually represents a static tunneling between two asymptotically flat regions when we give these Klein bottles vanishing relative velocity $v=0$ and is equivalent to extracting two Klein bottles, with geometric parameters given by Eqs. (2.2), (2.3), (2.5), and (2.6), from three-dimensional Euclidean space, and identifying the Klein bottle surfaces, so that when you enter the surface of, say, the right Klein bottle, you find yourself emerging from the surface of the left Klein bottle and vice versa. In Minkowski spacetime, the Klein bottlehole can then be obtained, identifying the two world nonorientable concentric tube pairs swept out by the two Klein bottles, with events at the same Lorentz time identified.

Converting this Klein bottlehole into time machine is very simple: one sets one of the nonorientable hole mouths in motion at a given speed realtive to the other mouth, identifying then the two Klein bottlehole's mouths to each other. We shall consider now the spacetime metric of the resulting accelerating Klein bottlehole. Let us assume the right mouth to be the mouth which is moving. Then, just outside the right asymptotic rest frame, the transformation of the Klein bottlehole coordinates into external, Lorentz coordinates with metric

$$ds^2 = -dT^2 + \sum_{\alpha=1}^{3} dX_{i\alpha}^2, \quad i=1,2,$$

can be given as

$$T_i = T_R + v\gamma l_i \sin\varphi_2, \quad X_{i3} = X_{3R} + \gamma l_i \sin\varphi_2, \quad (4.1)$$

$$X_{i1} = m_i(l_i)\sin\varphi_1, \quad X_{i2} = m_i(l_i)\cos\varphi_1, \quad (4.2)$$

where $v = dX_{3R}/dT_R$ is the velocity of the right mouth, $X_3 = X_{3R}(t)$, $T = T_R(t)$ is the world line of the mouth's center, $dt^2 = dT_R^2 - dX_{3R}^2$, and γ is the relativistic factor $\gamma = 1/\sqrt{1-v^2}$. It follows that just outside the left asymptotic rest frame, one should have the transformation

$$T = t, \quad X_{i3} = X_{3L} + l_i \sin\varphi_2, \quad (4.3)$$

with the expressions for X_{i1} and X_{i2} also given by Eq. (4.2). In Eq. (4.3), X_{3L} is the time-independent X_3 location of the left mouth's center of the Klein bottle. One can write then the metric inside the accelerating Klein bottlehole and outside but near its mouths as

$$\begin{aligned} ds^2 = &-e^{2\Phi}dt^2 + \theta(2\pi - \varphi_1)(\{-[1 + gl_1F(l_1)\sin\varphi_2]^2 + 1\} \\ &\times e^{2\Phi}dt^2 + c(1)dl_1^2 + d(1)Q(1)dr_1 d\varphi_1 + d\Omega_1^2) \\ &+ \theta(\varphi_1 - 2\pi)(\{-[1 + gl_2F(l_2)\sin\varphi_2]^2 + 1\}e^{2\Phi}dt^2 \\ &+ c(2)dl_2^2 + d(2)Q(2)dr_2 d\varphi_1 + d\Omega_2^2), \end{aligned} \quad (4.4)$$

where $g = \gamma^2 dv/dt$ is the acceleration of the right mouth and Φ is the same function as for the original static Klein bottlehole. The functions $F(l_i)$ are form factors that vanish on the left half of the hole where $l_i \leqslant 0$, rising monotonously from 0 to 1 as one moves rightward from the throat to the right mouth [2]. Also used to obtain metric (4.4) are the definitions, $dv = gdt/\gamma^2$, $dt = dT_R/\gamma$, and $d\gamma = vg\gamma dt$.

Metric (4.4) is a specialization to nonorientable symmetry from the metric used for accelerating toroidal ringholes [3]. Using for Eq. (4.4) the set of parameters $A_1 = C_1 \neq 0$, $B_1 = D_1 \neq 0$, $A_2 = C_2 = B_2 = D_2 = 0$, we in fact obtain the metric for an accelerating ringhole [3]. Moreover, with the additional transformations $a \to 0$, $\varphi_2 \to \theta + \pi/2$, $\varphi_1 \to \phi$, we finally get the metric used by Morris *et al.* for accelerating spherical wormholes [2], starting from Eq. (4.4).

At sufficiently late times, accelerating Klein bottleholes can generate CTCs by exactly the same causes as in Misner or accelerating wormhole and ringhole spaces [2,23]: on the left mouth the Lorentz time and the proper time coincide, but on the right mouth the latter time is relativistically dilated. When this proper time shift becomes larger than the separation between the hole mouths, then CTCs would appear. This happens once the so-called chronology (Cauchy) horizon is reached. Such a horizon is the onset of the nonchronal region and divides the spacetime into two parts with completely different causal properties. Like in Misner and accelerating wormhole and ringhole spaces, there will be two families of timelike geodesics in the chronal region of accelerating Klein bottlehole space, rightward geodesics and leftward geodesics, both possessing their own chronology horizons and nonchronal regions [2,3]. All the mouth's lensing actions produced in accelerating ringholes [3] are expected to occur in the present case as well, including the drastic changes of the geometry of the chronology horizon that originates, roughly speaking, a compact fountain and a light cone at one of the hole's mouths [8,24]. Thus, if you go through the Klein bottlehole along a given world line, then one of the above chronology horizons and its nonchronal region are destroyed. The chronology horizon is transformed into just a boundary for the future Cauchy development of the compact fountain, generated by null geodesics which are past directed, to asymptote and enter the fountain [1]. All the effects caused by this action are qualitatively similar to those caused in accelerating ringholes [3] and, therefore, the reader interested in more details on these effects is referred to Ref. [3].

Clearly, the acutest problem with the kind of time machines being considered arises from the instabilities that such spacetimes show when quantum vacuum polarization is taken into account. In order to investigate what is going on, let us consider the point-splitting regularized Hadamard two-point function for a quantized, massless conformally coupled scalar field propagating in the spacetime of an accelerating Klein bottlehole. For regions where the curvature nearly vanishes [25], the Hadamard function can now be written in the form

$$G_{reg,i}^{(1)\pm}(x,x') = \sum_{N=1}^{\infty} \frac{\xi}{4\pi^2 D}\left(\frac{b_i\xi}{2D}\right)^{N-1} \times \left(\frac{1}{\lambda_{Ni}^{\pm}(x,x')} + \frac{1}{\lambda_{Ni}^{\pm}(x',x)}\right), \quad (4.5)$$

where

$$\xi = \sqrt{\frac{1-v}{1+v}} < 1, \quad (4.6)$$

D is the spatial length of a geodesic that connects points x and x' by traversing once the Klein bottlehole, and

$$\lambda_{Ni}^{\pm}(x,x')=\xi^N\frac{\sigma_{Ni}^{\pm}}{\zeta_N}, \tag{4.7}$$

in which

$$\zeta_N=D\left(\frac{1-\xi^N}{1-\xi}\right) \tag{4.8}$$

and $\sigma_{Ni}^{\pm}$ is the Nth geodetic interval between x and x', for $(+)$ $2\pi-\varphi_2^c>\varphi_2>\varphi_2^c$ and $(-)-\varphi_2^c<\varphi_2<\varphi_2^c$, and $(i=1)$ $0\leqslant\varphi_1\leqslant 2\pi$ and $(i=2)$ $3\pi\geqslant\varphi_1\geqslant 2\pi$. Here $\lambda_{Ni}^{\pm}$ has been evaluated by means of a generalization [21] of the method originally used by Hiscock and Konkowski [25]. For the case of an accelerating Klein bottlehole space, the use of Fig. 1 and the covering space which distinguishes identified points in the original space [22] allows us to compute the displacements at fixed times T and T' of, respectively, copy 0 of x' and copy N of x from the covering-space throat location for $\varphi_1=0$ or $\varphi_1=2\pi$, i.e.,

$$\Delta\tilde{Y}_{0i}^{\pm}(x')=-(a_i\pm b_i)+m_i, \tag{4.9}$$

$$\Delta\tilde{Y}_{Ni}^{\pm}(x)=\xi^{-N}[(a_i\pm b_i)-m_i]. \tag{4.10}$$

Hence we get the corresponding geodetic intervals when the points x' and x are not on the symmetry axis of the Klein bottlehole:

$$\begin{aligned}\sigma_{Ni}^{\pm}&=\zeta_N[(\Delta\tilde{Y}_{Ni}^{\pm}\xi^N-T)\xi^{-N}-(\Delta\tilde{Y}_{0i}^{\pm}-T')]\\&=\zeta_N(\xi^{-N}\{\pm b_i[1\mp(-1)^i\cos\varphi_2]-T\}\\&\quad+\{\pm b_i[1\mp(-1)^i\cos\varphi_2]+T'\}).\end{aligned} \tag{4.11}$$

What is of most interest is the case when the points x' and x are also slightly off the throat in the Y direction. Then we obtain

$$\begin{aligned}\lambda_{Ni}^{\pm}(x,x')=&\pm 2b_i[1\pm(-1)^i\cos\varphi_2]\\&+(Y-T)-(Y'-T')\xi^N.\end{aligned} \tag{4.12}$$

In order to uncover the quantum instabilities that can take place in the accelerating Klein bottlehole, it is useful to introduce the concept of Nth polarized hypersurface H_N, i.e., that hypersurface which is formed by those events that join to themselves through closed null geodesics by traversing the Klein bottlehole N times [21]. Its interest arises from the fact that quantum vacuum polarization diverges on such hypersurfaces. Since Klein bottleholes are nothing but a topological generalization of Misner space, one should expect Nth polarized hypersurfaces to exist in the accelerating hole space with the symmetry of a Klein bottle. In fact, upon collapsing the points x' and x together in Eq. (4.12) it follows that there will be polarized hypersurfaces at times fixed by the condition $\sigma_{Ni}^{\pm}=0$, and hence $\lambda_{Ni}^{\pm}=0$, i.e., at times

$$T_{H_Ni}^{\pm}=\pm\frac{\xi^{-N}+1}{\xi^{-N}-1}b_i[1\mp(-1)^i\cos\varphi_2]. \tag{4.13}$$

There will be four chronology (Cauchy) horizons $H_i^{\pm}$, which appear as the limit as $N\to\infty$ of the times $T_{H_Ni}^{\pm}$ in accelerating Klein bottlehole space. They will, respectively, nest the corresponding polarized hypersurfaces defined at the times given by Eq. (4.13). On the symmetry axis where $\varphi_2=\pi$ and $\varphi_2=0$, all polarized hypersurfaces H_{Ni} occur at the same time only at $T=0$; away from this symmetry axis, one meets the polarizaed hypersurfaces one after another beginning with arbitrarily large N and ending at $N=1$, as T increases if we are in the region $2\pi-\varphi_2^c>\varphi_2>\varphi_2^c$ or as T decreases if $-\varphi_2^c<\varphi_2<\varphi_2^c$.

Each of the four different chronology horizons nests a set of polarized hypersurfaces. The nesting of hypersurfaces H_{N1} in the chronology horizon H_1^+, occurring at time

$$T_{H_1^+}=+b_1(1+\cos\varphi_2),$$

guarantees that an observer entering the region of CTCs will pass first through the chronology horizon H_1^+, and then successively through the H_{N_1}'s, at which hypersurfaces the observer would experience the strong peaks of vacuum polarization. The same behavior would also be expected for the chronology horizon H_2^+, which occurs at the different time

$$T_{H_2^+}=b_2(1-\cos\varphi_2).$$

In the case of the other two horizons occurring at times

$$T_{H_1^-}=-b_1(1-\cos\varphi_2)$$

and

$$T_{H_2^-}=-b_2(1+\cos\varphi_2),$$

the observer will first pass through the corresponding successive polarized hypersurfaces ($H_{N_1}^-$ and $H_{N_2}^-$, respectively) and then enter the given chronology horizon.

Anyway, we see that the kind of semiclassical instabilities which were present in wormholes and ringholes are also present in Klein bottleholes. We could, however, redefine the vacuum corresponding to the accelerating holes in Euclidean space to make it self-consistent [6]. With this new unique vacuum, the renormalized energy-momentum tensor should turn out to vanish everywhere, in principle so avoiding any instabilities of the accelerating Klein bottlehole or actually any of the time machines obtained by topologically generalizing Misner space. Nevertheless, according to Kay, Radzikowski, and Wald [13], this will only guarantee quantum stability on the regions just up to the chronology horizon, since such horizons lose their physical meaning also in the new vacuum, and the quantum divergence problem would still remain.

In spite of the failure to keep quantum-mechanically stable macroscopic time machines, one still could make it possible to avoid quantum instabilities in the above accelerating holes if we consider holes obtained as generalizations from a modified Misner space where the period of the closed spatial direction becomes time dependent and given by $2\pi T$ [7]. Using then automorphic fields [26] to compute the Hadamard function one obtains a quantization condition for time $T=(N+\alpha)T_0$, where $0\leqslant\alpha\leqslant\frac{1}{2}$ is the automorphic constant and T_0 is a minimum constant time whose most sensible value would probably be situated on the Planck scale. Translating into the language used above, it follows that the condition for the existence of the Nth polarized hypersurfaces on which quantum polarization of vacuum diverges should in this case imply

$$\frac{(1+\xi^N)b_i}{(1-\xi^N)(N+\alpha)}[1\mp(-1)^i\cos\varphi_2]=T_0=\mathrm{const},$$

which the system obviously cannot satisfy, unless ξ, b_i, φ_2, and N take on specific constant values. Therefore, no Nth polarized hypersurfaces could exist in any of the accelerating holes which are generalizations from this modified Misner space. However, on such generalizations only are possible CTCs with sizes of the order of the Planck time [7] and their chronology horizons will possess nonzero width, also on the Planck scale.

V. CONCLUSION

Using a convenient *Ansatz* for the geometric parameters that describe a Klein bottle, we have obtained exact solutions to the associated Einstein equations for two-dimensional sections in the vacuum case. These solutions possess apparent horizons at fixed valued of the two angular variables used to describe the Klein bottle. Starting with these solutions, we have constructed a spacetime which represents a Klein bottlehole tunneling that connects two asymptically flat large regions by shortcutting spacetime and found its embedding conditions in flat space. The latter conditions require that, at or near the throat, the embedding surface flare toward the two extreme values of the radius of the transversal section of the Klein bottle tube, and ultimately correspond to a rather complicated distribution of ordinary and exotic matter around the hole throat. The matter distribution allows, however, the existence of itineraries through the tunnel along which an observer could avoid finding regions with negative energy density, and gives rise to different lensing effects of the Klein bottlehole's mouths. We then constructed a time machine out of this spacetime hole and obtained its metric by allowing one of the hole's mouths to move relative to the other (see, however, [27]). This will allow that, at sufficiently late times, CTCs arise in some nonchronal region by relativistic time dilation.

There are four different chronology horizons in the resulting accelerating Klein bottlehole. Roughly speaking, the chronology horizons can be regarded to be like light cones developed from points of the original space. The four distinct horizons nest four correspondingly different classes of polarized hypersurfaces on which vacuum quantum fluctuations diverge, so making the time machine quantum-mechanically unstable. This result is inscapably obtained whenever our nonorientable spacetime construct can be regarded as a topological direct generalization from the usual Misner space, where nearly the same kind of instability also occurs. However, we have argued that if the nonorientable accelerating Klein bottlehole and, actually, any other such topological generalizations (e.g., accelerating wormholes and ringholes) are instead taken to be similar generalizations from the recently proposed modified Misner space [7] (i.e., Misner space with a time-dependent period of the closed spatial direction), then the calculation of the regularized two-point Hadamard function implies a quantization of time that ultimately prevents the existence of polarized hypersurfaces and, hence, leads to a quantum-mechanically stable time machine. The price to be paid for this is to have to renounce to the possibility of having time machines which would produce CTCs involving large time displacements. In fact, when time is quantized in very small steps T_0, the resulting CTCs involve only time intervals of order T_0 [7]. It is in this sense that stabilization of time machines does not induce any violation of semiclassical chronology protection conjecture [8]: quantization of time is simply not included in the conjecture.

Quantum spacetime foam can be thought to have a number of components, such as wormholes, virtual black holes [28], etc., among which quantum time machines inducing local violations of causality and orientability (i.e., accelarating Klein bottleholes arising as generalizations from modified Misner space) seem to be most necessary if the foam is defined in terms of minimal values of time and length at nearly the Planck scale [19]. Whether or not a future civilization will be able to extract, grow, and mantain such time machines out from the foam in such a way that the minimum time T_0 and, hence, CTCs be also scaled to large values is a question that only future developments might answer.

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